THE EQUALITY $I^2 = QI$ IN BUCHSBAUM RINGS WITH MULTIPLICITY TWO

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Abstract. Let A be a Buchsbaum local ring with the maximal ideal $\mathfrak{m}$ and let $\mathrm{e}(A)$ denote the multiplicity of A. Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}$. Then the equality $I^2 = QI$ holds true, if $\mathrm{e}(A) = 2$ and $\operatorname{depth} A > 0$. The assertion is no longer true, unless $\mathrm{e}(A) = 2$. Counterexamples are given.

1. Introduction.

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. Let Q be a parameter ideal in A and let $I = Q : \mathfrak{m}$. In this paper we are interested in the problem of when the equality $I^2 = QI$ holds true. This problem was completely solved by A. Corso, C. Huneke, C. Polini, and W. Vasconcelos [CHV, CP, CPV] in the case where A is a Cohen-Macaulay ring. When A is a Buchsbaum ring, partial answers only recently appeared in the authors' paper [GSa], supplying [Y1, Y2] and [GN] with ample examples of ideals I, for which the Rees algebras $\mathrm{R}(I) = \bigoplus_{n \geq 0} I^n$, the associated graded rings $\mathrm{G}(I) = \mathrm{R}(I)/I\mathrm{R}(I)$, and the fiber cones $\mathrm{F}(I) = \mathrm{R}(I)/\mathfrak{m}\mathrm{R}(I)$ are all Buchsbaum rings with certain specific graded local cohomology modules.

This research is a succession of [GSa] and the present purpose is to prove the following, in which $\mathrm{e}(A) = \mathrm{e}^0_{\mathfrak{m}}(A)$ denotes the multiplicity of A with respect to the maximal ideal $\mathfrak{m}$.

Theorem (1.1). *Let A be a Buchsbaum ring. Assume that $\mathrm{e}(A) = 2$ and $\operatorname{depth} A > 0$. Then the equality $I^2 = QI$ holds true for all parameter ideals Q in A, where $I = Q : \mathfrak{m}$.*

The readers may consult [G2] for the structure of Buchsbaum local rings A with $\mathrm{e}(A) = 2$. There is given in [G2, Sections 3,4] a complete list of equi-characteristic non-Cohen-Macaulay Buchsbaum complete local rings A with $\mathrm{e}(A) = 2$, $\operatorname{depth} A > 0$, and infinite residue class fields.

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In their remarkable papers [CP, CPV] A. Corso, C. Polini, and W. Vasconcelos proved that the equality $I^2 = QI$ holds true for every parameter ideal Q in A, if A is a Cohen-Macaulay local ring with $\mathrm{e}(A) \geq 2$. This is no more true in the Buchsbaum case. As is shown in [GSa, Theorem (4.8)], for every integer $e \geq 3$, there exists a Buchsbaum local ring A with $\dim A = 1$ and $\mathrm{e}(A) = e$ which contains a parameter ideal Q such that $I^e = QI^{e-1}$ but $I^{e-1} \neq QI^{e-2}$. Accordingly, without additional assumptions on Buchsbaum local rings A, no hope is left for the equality $I^2 = QI$, at least in the case where $\dim A = 1$ and $\mathrm{e}(A) \geq 3$. Our theorem (1.1) settles the case where $\mathrm{e}(A) = 2$ and $\operatorname{depth} A > 0$, providing a drastic break-through against the counter-examples of [GSa].

Before entering details, let us briefly note how this paper is organized. We shall prove Theorem (1.1) in Section 3. For the purpose we need some preliminaries, that we will summarize in Section 2. The counterexamples given by [GSa] are all of dimension 1, and according to Theorem (1.1), it might be natural to suspect that the equality $I^2 = QI$ holds true in higher dimensional cases of higher depth. The answer is, nevertheless, still negative. We shall construct examples, showing that for given integers $1 \leq d < m$, there exists a Buchsbaum local ring A with $\dim A = d$, $\operatorname{depth} A = d - 1$, and $\mathrm{e}(A) = 2m$, which contains a parameter ideal Q such that $I^3 = QI^2$ but $I^2 \neq QI$ (Theorem (4.5) and Proposition (4.7)).

2. Preliminaries.

The purpose of this section is to summarize some preliminary steps, which we need to prove Theorem (1.1). The result might have its own significance. In such a case we shall include a closer proof.

Here let us fix our standard notation. Otherwise specified, let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. For an ideal $\mathfrak{a}$ let $\mathfrak{a}^\sharp$ be the integral closure of $\mathfrak{a}$. Let $\ell_A(*)$ and $\mu_A(*)$ respectively denote the length and the number of generators. When A is a Cohen-Macaulay local ring, we denote by $\mathrm{r}(A)$ the Cohen-Macaulay type of A, that is

$$\mathrm{r}(A) = \ell_A(\mathrm{Ext}_A^d(A/\mathfrak{m}, A)).$$

Let $\mathrm{H}_\mathfrak{m}^i(*)$ ($i \in \mathbb{Z}$) be the local cohomology functors of A with respect to $\mathfrak{m}$. We denote by $\mathrm{e}(A) = \mathrm{e}_\mathfrak{m}^0(A)$ the multiplicity of A.

Let us begin with the following.

Lemma (2.1). *Assume that* $\mathrm{e}(A) \geq 2$. *Then* $\mu_A(I) = \ell_A(I/Q) + d$ *for every parameter ideal* Q *in* A, *where* $I = Q : \mathfrak{m}$.

Proof. By [GSa, Proposition (2.3)] Q is a minimal reduction of I, whence $\mathfrak{m}I = \mathfrak{m}Q$.

Thus

$$\mu_A(I) = \ell_A(I/\mathfrak{m}I) = \ell_A(I/Q) + \ell_A(Q/\mathfrak{m}Q) = \ell_A(I/Q) + d$$

as is claimed. $\square$

Proposition (2.2). *Suppose that A is a Cohen-Macaulay local ring with $d = \dim A \geq 1$ and let Q be a parameter ideal in A. Then*

$$\ell_A((0) :_{\mathfrak{m}/Q\mathfrak{m}} \mathfrak{m}) = \begin{cases} \mathrm{r}(A) + d & \text{if} \quad Q \neq Q^\sharp, \\ d & \text{if} \quad Q = Q^\sharp. \end{cases}$$

Proof. Since $\dim A \geq 1$, we have $Q\mathfrak{m} : \mathfrak{m} \subseteq \mathfrak{m}$, whence

$$\ell_A((0) :_{\mathfrak{m}/Q\mathfrak{m}} \mathfrak{m}) = \ell_A((Q\mathfrak{m} : \mathfrak{m})/Q\mathfrak{m}).$$

Let $I = Q : \mathfrak{m}$. Firstly, assume that $Q \neq Q^\sharp$. Then Q is a minimal reduction of I, because $I^2 = QI$ (cf. e.g., [GH, Proposition (3.4)]). Hence $\mathfrak{m}I = \mathfrak{m}Q$, so that we have $I \subseteq Q\mathfrak{m} : \mathfrak{m}$; thus $I = Q\mathfrak{m} : \mathfrak{m}$. Consequently,

$$\ell_A((0) :_{\mathfrak{m}/Q\mathfrak{m}} \mathfrak{m}) = \ell_A(I/\mathfrak{m}Q) = \ell_A(I/Q) + \ell_A(Q/\mathfrak{m}Q) = \mathrm{r}(A) + d.$$

Suppose that $Q = Q^\sharp$. Then A is a regular local ring which contains a regular system $a_1, a_2, \cdots, a_d$ of parameters such that $Q = (a_1, \cdots, a_{d-1}, a_d^q)$ for some $q \geq 1$ ([G3, Theorem (3.1)]). Hence $Q\mathfrak{m} : \mathfrak{m} = Q$, because $\mathfrak{m} = (a_1, a_2, \cdots, a_d)$ and $Q\mathfrak{m} : \mathfrak{m} \subseteq (a_1, \cdots, a_{d-1}, a_d^{q+1}) : a_d = Q$. Thus

$$\ell_A((0) :_{\mathfrak{m}/Q\mathfrak{m}} \mathfrak{m}) = \ell_A(Q/\mathfrak{m}Q) = d$$

as is claimed. $\square$

Let A be a Buchsbaum local ring with the Buchsbaum invariant $\mathrm{I}(A)$. Then all the local cohomology modules $\mathrm{H}^i_\mathfrak{m}(A)$ $(i \neq d)$ are killed by the maximal ideal $\mathfrak{m}$ and one has the equality

$$\mathrm{I}(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} h^i(A),$$

where $h^i(A) = \ell_A(\mathrm{H}^i_\mathfrak{m}(A))$ ([SV, Chap. I, Proposition 2.6]). Let

$$\mathrm{r}(A) = \sup_Q \ell_A((Q : \mathfrak{m})/Q)$$

where Q runs over parameter ideals in A and call it the Cohen-Macaulay type of A. We then have

$$\mathrm{r}(A) = \sum_{i=0}^{d-1} \binom{d}{i} h^i(A) + \mu_{\hat{A}}(\mathrm{K}_{\hat{A}}) \tag{2.3}$$

([GSu, Theorem (2.5)]), where $\mathrm{K}_{\hat{A}}$ denotes the canonical module of the $\mathfrak{m}$-adic completion $\widehat{A}$ of A. Consequently $\mathrm{r}(A) < \infty$.

Theorem (2.4). *Let B be a Gorenstein local ring with $d = \dim B \geq 2$. Let A be a subring of B. Assume that B is a module-finite extension of A and $\ell_A(B/A) = 1$. Then*

(1) *A is a Buchsbaum local ring and $\dim A = \mathrm{I}(A) = d$.*

(2) *The equality $I^2 = QI$ holds true for all parameter ideals Q in A, where $I = Q : \mathfrak{m}$.*

Proof. Since B is a module-finite extension of A, by Eakin-Nagata's theorem our ring A is a Noetherian local ring with $d = \dim\ A$. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the maximal ideals in A and B, respectively. We look at the exact sequence

$$0 \to A \xrightarrow{\iota} B \to A/\mathfrak{m} \to 0 \tag{2.5}$$

of A-modules, where ι denotes the inclusion map. Then applying functors $\mathrm{H}^i_{\mathfrak{m}}(*)$ to (2.5), we get that

$$\mathrm{H}^i_{\mathfrak{m}}(A) = (0)\ (i \neq 1, d) \ \text{ and } \ \mathrm{H}^1_{\mathfrak{m}}(A) \cong A/\mathfrak{m},$$

because $\mathrm{depth}_A B = d$. Hence A is a Buchsbaum ring with $\mathrm{I}(A) = d$ (cf. [SV, Chap. I, Proposition 2.12]). Notice that $\mathfrak{m}B = \mathfrak{m}$, because $\mathfrak{m}{\cdot}(B/A) = (0)$. Hence $\mathfrak{m}$ is an ideal in B. On the other hand, we naturally have by (2.5) the exact sequence

$$0 \to A/\mathfrak{m} \to B/\mathfrak{m}B \to A/\mathfrak{m} \to 0,$$

whence $\mu_A(B) = 2$. Let us write $B = A + At$ with $t \in B$. Then $t \notin A$. We have $\mathrm{r}(A) = d + 2$ by (2.3), since $\mathrm{K}_A = B$. Notice that $\mathrm{e}(A) \geq 2$, because A is not a regular local ring.

Now let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A and put $I = Q : \mathfrak{m}$. Then I is an ideal in B, since so is $\mathfrak{m}$. Thus $QB \subseteq I$. We need the following.

Claim (2.6). *$\ell_A(QB/Q) = d$.*

Proof of Claim (2.6). Since $B = A + At$, we have $QB = Q + \sum_{i=1}^d A{\cdot}a_it$. Let $\overline{a_it}$ denote the reduction of a_it mod Q. Then $QB/Q = \sum_{i=1}^d k{\cdot}\overline{a_it}$ $(k = A/\mathfrak{m})$. Let $\alpha_i \in A$ $(1 \leq i \leq d)$ and assume that $\sum_{i=1}^d \alpha_i(a_it) \in Q$. We write $\sum_{i=1}^d \alpha_i(a_it) = \sum_{i=1}^d \beta_i a_i$ with $\beta_i \in A$. Then $\sum_{i=1}^d a_i(\alpha_it - \beta_i) = 0$. Because $a_1, a_2, \cdots, a_d$ forms a B-regular sequence, $\alpha_it - \beta_i \in (a_j \mid j \neq i)B \subseteq A$, so that $\alpha_it \in A$. Hence $\alpha_i \in \mathfrak{m}$, because $t \notin A$. Thus the classes $\{\overline{a_it}\}_{1 \leq i \leq d}$ form a k-basis of QB/Q. Hence $\ell_A(QB/Q) = d$. $\square$

If $\ell_A(I/Q) = \mathrm{r}(A)$, then $I^2 = QI$ by [GSa, Theorem (3.9)]. Therefore to prove $I^2 = QI$, we may assume that $\ell_A(I/Q) \leq d{+}1$. Hence, either $\ell_A(I/Q) = d$, or $\ell_A(I/Q) = d{+}1$ (cf. Claim (2.6)). If $\ell_A(I/Q) = d$, then $I = QB$, so that $I^2 = QB{\cdot}IB = QI$. Assume that $\ell_A(I/Q) = d + 1$. Then $\ell_A(I/QB) = 1$. Therefore $\ell_B(I/QB) = 1$ and $\mathfrak{n}I \subseteq QB$. Hence $I = QB : \mathfrak{n}$, because $QB \subsetneq I \subseteq QB : \mathfrak{n}$ and B/QB is an Artinian Gorenstein local

ring. Accordingly, $I^2 = QB{\cdot}IB = QI$, if $QB \neq (QB)^\sharp$ in B (cf., e.g., [GH, Proposition (3.4)]). Suppose that $QB = (QB)^\sharp$ in B. Then, since $\mathrm{e}(A) \geq 2$, by [GSa, Proposition (2.3)] we have $I \subseteq Q^\sharp$. Hence $I \subseteq (QB)^\sharp = QB$ so that $I = QB$, which is impossible, because $\ell_A(I/QB) = 1$. Thus $QB \neq (QB)^\sharp$ in B and $I^2 = QI$, which completes the proof of Theorem (2.4). $\square$

The proof of the following result (2.7) is essentially the same as that of Theorem (2.4). Let us indicate a sketch only.

Proposition (2.7). *Let B be a Gorenstein local ring with the maximal ideal $\mathfrak{n}$ and $d = \dim B \geq 2$. Let A be a subring of B such that B is a finitely generated A-module. Assume that $A \subsetneq B$ and $\mathfrak{n} \subseteq A$. Then*

(1) *A is a Buchsbaum local ring with $\mathfrak{n}$ the maximal ideal and $\mathrm{I}(A) = d{\cdot}\ell_A(B/A)$.*
(2) *The equality $I^2 = QI$ holds true for all parameter ideals Q in A, where $I = Q : \mathfrak{m}$.*

Proof. Similarly as in the proof of Theorem (2.4), A is a Buchsbaum local ring with $\mathfrak{n}$ the maximal ideal, $\mathrm{H}^i_\mathfrak{m}(A) = (0)$ $(i \neq 1, d)$, and $\mathrm{H}^1_\mathfrak{m}(A) \cong B/A$. Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}$. Then $QB \subseteq I \subseteq QB : \mathfrak{n}$, since I is an ideal of B and $\mathfrak{m} = \mathfrak{n}$ in our case. Therefore, either $QB = I$, or $I = QB : \mathfrak{n}$, since B/QB is an Artinian Gorenstein local ring. We certainly have $I^2 = QI$ if $QB = I$. Assume that $I \neq QB$. Then $I = QB : \mathfrak{n}$ and $I^2 = QB{\cdot}IB = QI$, because $QB \neq (QB)^\#$ in B for the same reason as in the proof of Theorem (2.4). $\square$

Before closing this section let us note one example satisfying the hypothesis of Proposition (2.7).

Example (2.8). Let K/k be a finite extension of fields and assume that $\delta = [K : k] \geq 2$. Let $n = \delta - 1$ and choose a k-basis $\{\theta_0 = 1, \theta_1, \cdots, \theta_n\}$ of K. Let $d \geq 2$ be an integer and let $B = K[[X_1, X_2, \cdots, X_d]]$ be the formal power series ring over K. Let

$$A = k[[\theta_i X_j \mid 0 \leq i \leq n, 1 \leq j \leq d]].$$

Then B is a module-finite extension of A such that the maximal ideal $\mathfrak{n}$ of B coincides with that of A, $\ell_A(B/A) = n$, and $\mathrm{e}(A) = \delta$. Hence by Proposition (2.7) A is a Buchsbaum local ring, in which the equality $I^2 = QI$ holds true for all parameter ideals Q, where $I = Q : \mathfrak{m}$.

Proof. Let $\mathfrak{m}$ be the maximal ideal in A, that is $\mathfrak{m} = (\theta_i X_j \mid 0 \leq i \leq n, 1 \leq j \leq d)A$. Then $\mathfrak{m}B = \mathfrak{n}$ whence $B = \sum_{i=0}^n A\theta_i$. Let $b \in B$ and write $b = \sum_{i=0}^n a_i\theta_i$ with $a_i \in A$. Then, since $bX_j = \sum_{i=0}^n a_i(\theta_i X_j) \in \mathfrak{m}$ for all $1 \leq j \leq d$, we get $\mathfrak{n} \subseteq \mathfrak{m}$. Hence $\mathfrak{m} = \mathfrak{n}$. We have

$$\mu_A(B) = \ell_A(B/\mathfrak{m}B) = \ell_A(B/\mathfrak{n}) = [K : k] = \delta \geq 2.$$

Because $\ell_A(B/A) = \ell_A(B/\mathfrak{n}) - \ell_A(A/\mathfrak{m})$, we get $\ell_A(B/A) = \delta - 1 = n$. Let $\mathfrak{q} = (X_1, X_2, \cdots, X_d)A$. Then since $\mathfrak{m}^2 = \mathfrak{q}\mathfrak{m}$, the ideal $\mathfrak{q}$ is a minimal reduction of $\mathfrak{m}$, so that we have

$$\mathrm{e}(A) = \mathrm{e}^0_{\mathfrak{q}}(A) = \mathrm{e}^0_{\mathfrak{q}}(B) = \ell_A(B/\mathfrak{q}B) = \ell_A(B/\mathfrak{n}) = \delta,$$

as is claimed. □

3. Proof of Theorem (1.1)

Let A be a Noetherian local ring with $d = \dim A \geq 2$. Suppose that A is a reduced ring with $\#\operatorname{Ass} A = 2$, say $\operatorname{Ass} A = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. We furthermore assume that $A/\mathfrak{p}_i$ is a regular local ring with $\dim A/\mathfrak{p}_i = d$ $(i = 1, 2)$ and $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{m}$. Hence $\mathfrak{m} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, because $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$. With this notation and assumption we have the following.

Theorem (3.1). *$I^2 = QI$ for all parameter ideals Q in A, where $I = Q : \mathfrak{m}$.*

Proof. We look at the exact sequences

$$(3.2) \qquad 0 \to A \xrightarrow{\iota} A/\mathfrak{p}_1 \oplus A/\mathfrak{p}_2 \to A/\mathfrak{m} \to 0 \quad \text{and}$$

$$(3.3) \qquad 0 \to \mathfrak{p}_i \to A \to A/\mathfrak{p}_i \to 0$$

of A-modules $(i = 1, 2)$, where $\iota(a) = (a \mod \mathfrak{p}_1, a \mod \mathfrak{p}_2)$ for all $a \in A$. Then, applying functors $\mathrm{H}^i_{\mathfrak{m}}(*)$ to (3.2), we get $\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ $(i \neq 1, d)$ and $\mathrm{H}^1_{\mathfrak{m}}(A) \cong A/\mathfrak{m}$. Hence A is a Buchsbaum local ring with $\mathrm{I}(A) = d$ and $\mathrm{e}(A) = 2$. We have $\mathrm{r}(A) = d + 2$ by (2.3), because $\mathrm{K}_A = A/\mathfrak{p}_1 \oplus A/\mathfrak{p}_2$. Let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A and put $I = Q : \mathfrak{m}$. Then $\ell_A(I/Q) \leq r(A) = d+2$. We may assume $\ell_A(I/Q) \leq d+1$ (cf. [GSa, Theorem (3.9)]). Let $A_i = A/\mathfrak{p}_i$ and $\mathfrak{m}_i = \mathfrak{m}/\mathfrak{p}_i$ $(i = 1, 2)$. We write $a_j = \ell_j + m_j$ $(1 \leq j \leq d)$ with $\ell_j \in \mathfrak{p}_1$ and $m_j \in \mathfrak{p}_2$.

Firstly we consider the case $QA_2 \neq (QA_2)^\#$ in A_2. Let $\varepsilon : A \to A_2$ be the canonical epimorphism. Then $\varepsilon(\mathfrak{p}_1) = \mathfrak{m}_2$ and $\mathfrak{p}_1 \cong \mathfrak{m}_2$ via ε, because $\mathfrak{m} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. Hence

$$(3.4) \qquad \ell_A((0) :_{\mathfrak{p}_1/Q\mathfrak{p}_1} \mathfrak{m}) = \ell_A((0) :_{\mathfrak{m}_2/Q\mathfrak{m}_2} \mathfrak{m}) = \ell_{A_2}((0) :_{\mathfrak{m}_2/Q\mathfrak{m}_2} \mathfrak{m}_2) = d + 1$$

by Proposition (2.2). Let $\overline{*}$ denote the reduction mod $\mathfrak{p}_2$. Then since A_2/QA_2 is an Artinian Gorenstein local ring and since $Q\mathfrak{m}_2 : \mathfrak{m}_2 = QA_2 : \mathfrak{m}_2 \subseteq \mathfrak{m}_2$ (cf. Proof of Proposition (2.2)), the the ideal $Q\mathfrak{m}_2 : \mathfrak{m}_2$ of A_2 is generated by $\{\overline{\ell_j}\}_{1 \leq j \leq d}$ together with one more element, say $\overline{\eta}$ $(\eta \in \mathfrak{p}_1)$. Hence the $A/\mathfrak{m}$-space $(0) :_{\mathfrak{p}_1/Q\mathfrak{p}_1} \mathfrak{m}$ is spanned by $\{\ell_j \mod Q\mathfrak{p}_1\}_{1 \leq j \leq d}$ and $\eta \mod Q\mathfrak{p}_1$, because $\mathfrak{p}_1 \cong \mathfrak{m}_2$ via ε. Now look at the exact sequence

$$(3.5) \qquad 0 \to \mathfrak{p}_i/Q\mathfrak{p}_i \to A/Q \xrightarrow{\varphi_i} A_i/QA_i \to 0 \quad (i = 1, 2)$$

of A-modules induced from (3.3) (notice that $a_1, a_2, \cdots, a_d$ is an A_i-regular sequence). Then, considering the socles of the terms in (3.5) with $i = 1$, we get

$$I = Q + (\ell_1, \ell_2, \cdots, \ell_d) + (\eta),$$

because $\ell_A((0) :_{\mathfrak{p}_1/Q\mathfrak{p}_1} \mathfrak{m}) = d + 1$ by (3.4) and $\ell_A(I/Q) \leq d + 1$ by our standard assumption. Hence $I^2 = QI + (\eta^2)$, because

$$\ell_i\ell_j = (\ell_i + m_i)\ell_j = a_i\ell_j \quad \text{and} \quad \ell_i\eta = (\ell_i + m_i)\eta = a_i\eta$$

for all $1 \leq i, j \leq d$ (recall that $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$). On the other hand, since $QA_2 \neq (QA_2)^{\#}$, we get $(QA_2 : \mathfrak{m}_2)^2 = QA_2 \cdot (QA_2 : \mathfrak{m}_2)$ (cf., e.g., [GH, Proposition (3.4)]). Hence $\overline{\eta}^2 \in (\overline{\ell_1}, \overline{\ell_2}, \cdots, \overline{\ell_d}) {\cdot} [(\overline{\ell_1}, \overline{\ell_2}, \cdots, \overline{\ell_d}) + (\overline{\eta})]$ so that

$$\eta^2 \in (\ell_1, \ell_2, \cdots, \ell_d){\cdot}[(\ell_1, \ell_2, \cdots, \ell_d) + (\eta)] + \mathfrak{p}_2. \tag{3.6}$$

Because $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$ and $(\ell_1, \ell_2, \cdots, \ell_d) + (\eta) \subseteq \mathfrak{p}_1$, by (3.6) we readily get that

$$\eta^2 \in (\ell_1, \ell_2, \cdots, \ell_d){\cdot}[(\ell_1, \ell_2, \cdots, \ell_d) + (\eta)] \subseteq QI.$$

Hence $I^2 = QI$, since $I^2 = QI + (\eta^2)$. We get, by the symmetry, that $I^2 = QI$ also in the case where $QA_1 \neq (QA_1)^{\#}$.

We now consider the case $QA_i = (QA_i)^{\#}$ for $i = 1, 2$. Then thanks to the exact sequence (3.5) with $i = 1$, we have $\ell_A(I/Q) \geq d$, because

$$\ell_A((0) :_{\mathfrak{p}_1/Q\mathfrak{p}_1} \mathfrak{m}) = \ell_{A_2}((0) :_{\mathfrak{m}_2/Q\mathfrak{m}_2} \mathfrak{m}_2) = d$$

by Proposition (2.2). We actually have the following .

Claim (3.7). *$\ell_A(I/Q) = d$.*

Proof of Claim (3.7). Suppose that $\ell_A(I/Q) \neq d$. Then $\ell_A(I/Q) = d+1$. Choose a regular system $\overline{c_1}, \overline{c_2}, \cdots, \overline{c_d}$ ($c_i \in \mathfrak{p}_1$) of parameter for A_2 so that $QA_2 = (\overline{c_1}, \cdots, \overline{c_{d-1}}, \overline{c_d}^q)$ for some $q > 0$ (cf. [G3, Theorem (3.1)]). Hence

$$QA_2 : \mathfrak{m}_2 = (\overline{c_1}, \cdots, \overline{c_{d-1}}, \overline{c_d}^{q-1}) = QA_2 + (\overline{c_d}^{q-1}).$$

We have

$$(\ell_1, \ell_2, \cdots, \ell_d) = (c_1, \cdots, c_{d-1}, c_d^q), \tag{3.8}$$

because $QA_2 = (\overline{\ell_1}, \overline{\ell_2}, \cdots, \overline{\ell_d}) = (\overline{c_1}, \cdots, \overline{c_{d-1}}, \overline{c_d}^q)$ and $\mathfrak{p}_1 \cong \mathfrak{m}_2$ via ε. Notice that

$$\ell_A((0) :_{\mathfrak{p}_2/Q\mathfrak{p}_2} \mathfrak{m}) = \ell_{A_1}((0) :_{\mathfrak{m}_1/Q\mathfrak{m}_1} \mathfrak{m}_1) = d$$

and $(0):_{\mathfrak{m}_1/Q\mathfrak{m}_1}\mathfrak{m}_1 = QA_1/Q\mathfrak{m}_1$ (cf. Proposition (2.2) and its proof). Hence the $A/\mathfrak{m}$-space $(0):_{\mathfrak{p}_2/Q\mathfrak{p}_2}\mathfrak{m}$ is spanned by $\{m_j \mod Q\mathfrak{p}_2\}_{1\le j\le d}$. We now look at the exact sequence (3.5) with $i=2$. Then since $\ell_A((0):_{\mathfrak{p}_2/Q\mathfrak{p}_2}\mathfrak{m}) = d$ and $\ell_A(I/Q) = d+1$, the canonical epimorphism $\varphi_2 : A/Q \to A_2/QA_2$ cannot be zero on the socles, so that we have $IA_2 = QA_2 : \mathfrak{m}_2$. Hence $IA_2 = QA_2 + (\overline{c_d}^{q-1})$. Choose $\eta \in I$ so that $\overline{\eta} = \overline{c_d}^{q-1}$ and write $\eta = c_d^{q-1} + \delta + \rho$ with $\delta \in Q$ and $\rho \in \mathfrak{p}_2$. Then thanks to the exact sequence (3.5) with $i=2$, we get

$$\begin{aligned} I &= Q + (m_1, m_2, \cdots, m_d) + (\eta) \\ &= Q + (m_1, m_2, \cdots, m_d) + (c_d^{q-1} + \delta + \rho), \end{aligned}$$

because $(0):_{\mathfrak{p}_2/Q\mathfrak{p}_2}\mathfrak{m}$ is spanned by $\{m_j \mod Q\mathfrak{p}_2\}_{1\le j\le d}$. Consequently we have

$$\begin{aligned} I &= Q + (m_1, m_2, \cdots, m_d) + (c_d^{q-1} + \rho) && \text{(since } \delta \in Q) \\ &= (\ell_1, \ell_2, \cdots, \ell_d) + (m_1, m_2, \cdots, m_d) + (c_d^{q-1} + \rho) \\ &= (c_1, \cdots, c_{d-1}, c_d^q) + (m_1, m_2, \cdots, m_d) + (c_d^{q-1} + \rho) && \text{(by (3.8))} \\ &= (c_1, \cdots, c_{d-1}) + (m_1, m_2, \cdots, m_d) + (c_d^{q-1} + \rho) && \text{(since } c_d^q = c_d(c_d^{q-1} + \rho)), \end{aligned}$$

which is impossible, because $\mu_A(I) = \ell_A(I/Q) = 2d+1$ by Lemma (2.1). Thus $\ell_A(I/Q) = d$. $\square$

Therefore, in the exact sequence (3.5) with $i=2$, the socle I/Q of A/Q coincides with the image of $(0):_{\mathfrak{p}_2/Q\mathfrak{p}_2}\mathfrak{m}$, because $\ell_A((0):_{\mathfrak{p}_2/Q\mathfrak{p}_2}\mathfrak{m}) = d$; that is

$$I = Q + (m_1, m_2, \cdots, m_d).$$

Thus $I^2 = QI$, because $m_i m_j = (\ell_i + m_i)m_j \in QI$ for all $1 \le i, j \le d$. This completes the proof of Theorem (3.1). $\square$

We are now ready to prove Theorem(1.1).

Proof of Theorem (1.1). Passing to the ring $A[X]_{\mathfrak{m}A[X]}$ where X is an indeterminate over A and then passing to the completion, we may assume that A is a complete local ring with the infinite residue class field. Thanks to [CP, Theorem 2.2], we may assume that A is not a Cohen-Macaulay ring. Hence $d \ge 2$ and so by [G2, Theorem 1.1]

$$\mathrm{H}^i_{\mathfrak{m}}(A) = (0) \ (i \ne 1, d) \ \text{ and } \ \mathrm{H}^1_{\mathfrak{m}}(A) \cong A/\mathfrak{m}.$$

Let B be the Cohen-Macaulayfication of A, that is the intermediate ring $A \subseteq B \subseteq Q(A)$ where $Q(A)$ denotes the total quotient ring of A, such that B is a module-finite extension of A, $\operatorname{depth}_A B = d$, and $\mathfrak{m}B = \mathfrak{m}$ (cf. [G1, Theorem (1.1)]). Then $\mathrm{H}^1_{\mathfrak{m}}(A) \cong B/A$ and hence $\ell_A(B/A) = 1$. Now there are two cases. One is the case where B is a local ring.

The other one is the case where B is not a local ring. Firstly, suppose that B is a local ring and choose a minimal reduction $\mathfrak{q}$ of $\mathfrak{m}$. Then $\ell_A(B/\mathfrak{q}B) = 2$, because

$$\ell_A(B/\mathfrak{q}B) = \mathrm{e}^0_{\mathfrak{q}}(B) = \mathrm{e}^0_{\mathfrak{q}}(A) = \mathrm{e}(A).$$

Consequently $[B/\mathfrak{n} : A/\mathfrak{m}]{\cdot}\ell_B(B/\mathfrak{q}B) = 2$. If $A/\mathfrak{m} \neq B/\mathfrak{n}$, then $\ell_B(B/\mathfrak{q}B) = 1$ so that $\mathfrak{n} = \mathfrak{q}B$. Hence B is a regular local ring with $\mathfrak{n} = \mathfrak{m}$ and the assertion follows from Proposition (2.7). If $A/\mathfrak{m} = B/\mathfrak{n}$, then $\ell_B(B/\mathfrak{q}B) = 2$. Hence $\ell_B(\mathfrak{n}/\mathfrak{q}B) = 1$ and so B is a Gorenstein ring, because the ring $B/\mathfrak{q}B$ is Artinian and Gorenstein. The assertion now follows from Theorem (2.3), since $\ell_A(B/A) = 1$.

Assume that B is not a local ring. Then the proof of [G2, Proposition 4.2] still works in our context to show that A is a reduced ring with $\sharp \operatorname{Ass} A = 2$, say $\operatorname{Ass} A = \{\mathfrak{p}_1, \mathfrak{p}_2\}$, such that $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{m}$, and $A/\mathfrak{p}_i$ is a regular local ring with $d = \dim A/\mathfrak{p}_i$ for $i = 1, 2$. Hence the equality $I^2 = QI$ follows from Theorem (3.1). This completes the proof of Theorem (1.1). $\square$

4. Examples.

In this section we shall construct examples, showing that the equality $I^2 = QI$ fails in general to hold, even though A is a Buchsbaum local ring with sufficiently large depth and multiplicity.

Let k be a field and let $1 \leq d < m$ be integers. Let $S = k[X_1, \cdots, X_m, V, A_1, \cdots, A_d]$ be the polynomial ring with $m + d + 1$ variables over k and put

$$\mathfrak{a} = (X_i \mid 1 \leq i \leq m-1)^2 + (X_m^2) + (X_iV \mid 1 \leq i \leq m) + (V^2 - \sum_{i=1}^{d} A_iX_i).$$

We regard S as a $\mathbb{Z}$-graded ring such that $S_0 = k$ and $X_i, V, A_j \in S_1$ for all $1 \leq i \leq m$ and $1 \leq j \leq d$. We put

$$R = S/\mathfrak{a}, M = R_+, B = R_M, \text{ and } \mathfrak{m} = MB.$$

Then $\dim R = d$, because $\sqrt{\mathfrak{a}} = (X_i \mid 1 \leq i \leq m) + (V)$. Let x_i, v, and a_j denote respectively the reduction of X_i, V, and A_j mod $\mathfrak{a}$. We put $Q = (a_1, a_2, \cdots, a_d)$ and $\mathfrak{p} = (x_1, x_2, \cdots, x_m) + (v)$. Hence $M = Q + \mathfrak{p}$ and $M^3 = Q\mathfrak{p}^2$, since $\mathfrak{p}^3 = (0)$. Consequently, $\mathfrak{q} = QB$ is a minimal reduction of the maximal ideal $\mathfrak{m}$ in B.

Let us begin with the following.

Lemma (4.1). *$\ell_B(B/\mathfrak{q}) = 2m + 1$, $\mathrm{e}^0_{\mathfrak{m}}(B) = 2m$, and B is not a Cohen-Macaulay ring.*

Proof. Since $R/Q \cong k[X_1, X_2, \cdots, X_m, V]/\mathfrak{b}$ where

$$\mathfrak{b} = (X_i \mid 1 \leq i \leq m-1)^2 + (X_m^2, V^2) + (X_iV \mid 1 \leq i \leq m),$$

we have $\dim_k R/Q = 2m+1$ and

$$Q : M = Q + (x_i x_m \mid 1 \le i \le m-1) + (v).$$

Hence $\ell_B(B/\mathfrak{q}) = 2m+1$. We put $P = (X_i \mid 1 \le i \le m) + (V)$; hence $\mathfrak{p} = P/\mathfrak{a}$. Then $\operatorname{Min} B = \{\mathfrak{p}\}$ and $B/\mathfrak{p}$ is a regular local ring, so that we have $\mathrm{e}(B) = \ell_{B_\mathfrak{p}}(B_\mathfrak{p})$. Let $\widetilde{S} = S[\frac{1}{A_1}]$ and $\widetilde{k} = k[A_1, \frac{1}{A_1}]$. Then

$$\widetilde{S} = \widetilde{k}[X_1', \cdots, X_m', V', A_2', \cdots, A_d'],$$

where $X_i' = \frac{X_i}{A_1}$, $V' = \frac{V}{A_1}$, and $A_j' = \frac{A_j}{A_1}$ ($1 \le i \le m$, $1 \le j \le d$). The elements $\{X_i'\}_{1 \le i \le m}$, V', and $\{A_j'\}_{2 \le j \le d}$ are algebraically independent over $\widetilde{k}$. We have

$$\mathfrak{a}\widetilde{S} = (X_i' \mid 1 \le i \le m-1)^2 + ({X_m'}^2) + (X_i'V' \mid 1 \le i \le m) + ({V'}^2 - \sum_{i=1}^{d} A_i'X_i')$$

and $P\widetilde{S} = (X_i' \mid 1 \le i \le m) + (V')$. Hence $X_1' - ({V'}^2 - \sum_{i=2}^d A_i'X_i') \in \mathfrak{a}\widetilde{S}$. Let $T = \widetilde{k}[X_2', \cdots, X_m', V', A_2', \cdots, A_d']$ and we identify $T = \widetilde{S}/(X_1' - ({V'}^2 - \sum_{i=2}^d A_i'X_i'))$. Then

$$\begin{aligned}
\mathfrak{a}T &= ({V'}^2 - \sum_{i=2}^{d} A_i'X_i', \{X_i'\}_{2 \le i \le m-1})^2 + ({X_m'}^2) + \left(({V'}^2 - \sum_{i=2}^{d} A_i'X_i')\cdot V'\right) \\
&\quad + (X_i'V' \mid 2 \le i \le m) \\
&= ({V'}^2, \{X_i'\}_{2 \le i \le m-1})^2 + ({X_m'}^2) + ({V'}^3) + (X_i'V' \mid 2 \le i \le m) \quad \text{(since } d < m) \\
&= (X_i' \mid 2 \le i \le m-1)^2 + ({X_m'}^2) + ({V'}^3) + (X_i'V' \mid 2 \le i \le m),
\end{aligned}$$

and $PT = (X_i' \mid 2 \le i \le m) + (V')$. Therefore $\ell_{B_\mathfrak{p}}(B_\mathfrak{p}) = \ell_{S_p}(S_p/\mathfrak{a}S_p) = \ell_U(U)$, where

$$\begin{aligned}
U &= k(A_1', A_2', \cdots, A_d')[X_2', \cdots, X_m', V']/\mathfrak{b} \quad \text{and} \\
\mathfrak{b} &= (X_i' \mid 2 \le i \le m-1)^2 + ({X_m'}^2) + ({V'}^3) + (X_i'V' \mid 2 \le i \le m).
\end{aligned}$$

Consequently, $\mathrm{e}(B) = \ell_U(U) = 2m < \ell_B(B/\mathfrak{q}) = 2m+1$, whence B is not a Cohen-Macaulay ring. $\square$

Let $\underline{a}^2$ denote the sequence $a_1^2, a_2^2, \cdots, a_d^2$ and let $\mathrm{e}^0_{(\underline{a}^2)B}(B)$ denote the multiplicity of B with respect to the parameter ideal $(\underline{a}^2)B = (a_1^2, a_2^2, \cdots, a_d^2)B$. We then have the following.

Proposition (4.2). $\ell_B(B/(\underline{a}^2)B) - \mathrm{e}^0_{(\underline{a}^2)B}(B) = 1.$

Proof. Since B is not a Cohen-Macaulay ring, $\ell_B(B/(\underline{a}^2)B) - \mathrm{e}^0_{(\underline{a}^2)B}(B) > 0$. It suffices to show $\ell_B(B/(\underline{a}^2)B) - \mathrm{e}^0_{(\underline{a}^2)B}(B) \leq 1$. Let

$$\mathfrak{c} = (A_i^2 \mid 1 \leq i \leq d) + (X_i^2 \mid 1 \leq i \leq m) + (V^2 - \sum_{i=1}^{d} A_i X_i)$$

and put $C = S/\mathfrak{c}$ and $D = S/\left(\mathfrak{a} + (A_i^2 \mid 1 \leq i \leq d)\right)$. Then D is a homomorphic image of C. The ring C is a complete intersection with $\dim_k C = 2^{d+m+1}$. Let x_i, v, and a_j denote, for the moment, the reduction of X_i, V, and A_j mod $\mathfrak{a} + (A_i^2 \mid 1 \leq i \leq d)$. Let $\Lambda = \{1, 2, \cdots d\}$ and $\Gamma = \{1, 2, \cdots, m\}$. For given subsets $I \subseteq \Lambda$ and $J \subseteq \Gamma$ we put

$$a_I = \prod_{i \in I} a_i \quad \text{and} \quad x_J = \prod_{j \in J} x_j.$$

Then the elements $\{a_I x_J\}_{I \subset \Lambda, J \subset \Gamma}$ and $\{a_I x_J v\}_{I \subset \Lambda, J \subset \Gamma}$ span the k-space D, because their preimages in C form a k-bases of C. Notice that $a_I x_J v = 0$ if $J \neq \emptyset$ and that $J \subseteq \{i, m\}$ for some $1 \leq i \leq m-1$ if $x_J \neq 0$. Hence the k-space D is actually spanned by the following $2^d(2m+1)$ elements

$$a_I,\ x_i a_I,\ x_m a_I,\ x_i x_m a_I, \text{ and } a_I v \quad \text{with} \quad I \subseteq \Lambda, 1 \leq i \leq m-1. \tag{4.3}$$

Let $1 \leq i \leq d$ and $K \subseteq \Lambda$. Assume that $i \notin K$ but $\{1, \cdots, i-1\} \subseteq K$. Then since $(\sum_{i=1}^{d} a_i x_i)(x_m a_K) = v^2 x_m a_K = 0$, we have

$$\sum_{1 \leq j < i} (a_j x_j)(x_m a_K) + (a_i x_i)(x_m a_K) + \sum_{i < j \leq d} (a_j x_j)(x_m a_K) = 0. \tag{4.4}$$

Notice that $\sum_{1 \leq j < i} (a_j x_j)(x_m a_K) = 0$, since $(a_\ell x_\ell)(x_m a_K) = (a_\ell a_K)(x_\ell x_m) = 0$ for all $\ell \in K$. If $i < j \leq d$ and $j \notin K$, then $(a_j x_j)(x_m a_K) = (x_j x_m) a_{K \cup \{j\}}$. Consequently by (4.4) we have the following expression

$$x_i x_m a_{K \cup \{i\}} = (a_i x_i)(x_m a_K) = - \sum_{i < j \leq d, j \notin K} (x_j x_m) a_{K \cup \{j\}}$$

of $x_i x_m a_{K \cup \{i\}}$. Hence, letting $K = I \setminus \{i\}$, it follows from this expression that for all $1 \leq i \leq d$, the set $\{x_i x_m a_I\}_{\{1, \cdots, i\} \subseteq I \subseteq \Lambda}$ is contained in the k-subspace of D spanned by $\{x_j x_m a_J\}_{i < j \leq d, J \subseteq \Lambda}$. Therefore, in order to span the whole k-space D, for each $1 \leq i \leq d$ the elements $\{x_i x_m a_I \mid \{1, 2, \cdots, i\} \subseteq I \subseteq \Lambda\}$ can be deleted from the system of generators given by (4.3), so that we have

$$\begin{aligned}
\ell_B(B/(\underline{a}^2)B) = \dim_k D &\leq 2^d(2m+1) - \sum_{i=1}^{d} 2^{d-i} \\
&= 2^d(2m+1) - (2^d - 1) \\
&= 2^{d+1} m + 1 \\
&= 2^d \mathrm{e}^0_{(\underline{a})B}(B) + 1 \\
&= \mathrm{e}^0_{(\underline{a}^2)B}(B) + 1
\end{aligned}$$

as is claimed. $\square$

By Proposition (4.2) we get

$$\ell_B(B/(\underline{a}^2)B) - \mathrm{e}^0_{(\underline{a}^2)B}(B) = \ell_B(B/\mathfrak{q}) - \mathrm{e}^0_{\mathfrak{q}}(B) = 1.$$

Hence the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(B)$ $(i \neq d)$ are finitely generated B-modules and $\sum_{i=0}^{d-1} \binom{d-1}{i} h^i(B) = 1$ (cf. [SV, Appendix, Theorem and Definition 17]). Accordingly, either $\operatorname{depth} B = 0$, or $\operatorname{depth} B = d-1$. We have that $h^0(B) = 1$ and $h^i(B) = 0$ $(1 \leq i \leq d-1)$ if $\operatorname{depth} B = 0$, and that $h^{d-1}(B) = 1$ if $\operatorname{depth} B = d-1$. In any case, $\mathrm{H}^i_{\mathfrak{m}}(B) = (0)$ for all $i \neq t, d$, and $\mathrm{H}^t_{\mathfrak{m}}(B) \cong B/\mathfrak{m}$, where $t = \operatorname{depth} B$. Thus B is a Buchsbaum ring (cf. [SV, Chap. I, Proposition 2.6]). We actually have the following.

Theorem (4.5). $\mathrm{H}^{d-1}_M(R) \cong (R/M)(d-3)$.

Proof. (1) $(d = 1)$. Use the fact that $0 \neq x_1x_m \in (0) : M$.

(2) $(d = 2)$. Assume that $\operatorname{depth} B = 0$. Then applying functors $\mathrm{H}^i_M(*)$ to the exact sequence

$$0 \to \mathrm{H}^0_M(R) \to R(-1) \overset{a_2}{\to} R \to R/a_2R \to 0, \tag{4.6}$$

we get a natural isomorphism $\mathrm{H}^0_M(R) \cong \mathrm{H}^0_M(R/a_2R)$. We apply the result of the case where $d = 1$ to the ring R/a_2R and choose $0 \neq \varphi \in R_2$ such that $M\varphi = (0)$ and $\varphi \equiv x_1x_m \bmod a_2R$. Let $\varphi = x_1x_m + a_2\psi$ with $\psi \in R_1$. Then $x_1x_ma_2 + a_2^2\psi = 0$, that is $X_1X_mA_2 + A_2^2\xi \ \in \mathfrak{a}$ for some $\xi \in S_1$, which is impossible. Hence $\operatorname{depth} B = 1$ and by (4.6) we get an isomorphism $\mathrm{H}^0_M(R/a_2R) \cong \mathrm{H}^1_M(R)(-1)$. Thus $\mathrm{H}^1_M(R) \cong (R/M)(-1)$, because $\mathrm{H}^0_M(R/a_2R) \cong (R/M)(-2)$.

(3) $(d \geq 3)$. We may assume that our assertion holds true for $d-1$. Then $\operatorname{depth} B = d-1$, because

$$h^0(B/a_dB) = h^0(B) + h^1(B)$$

and $h^0(B/a_dB) = 0$ by the hypothesis on d. Hence a_d is a non-zerodivisor of R, so that we have

$$\mathrm{H}^{d-2}_M(R/a_dR) \cong \mathrm{H}^{d-1}_M(R)(-1).$$

Thus $\mathrm{H}^{d-1}_M(R) \cong (R/M)(d-3)$, because $\mathrm{H}^{d-2}_M(R/a_dR) \cong (R/M)(d-4)$. $\square$

Let $J = Q : M$ and $I = \mathfrak{q} : \mathfrak{m}$ $(= JB)$. We then have the following.

Proposition (4.7). *$I^2 \neq \mathfrak{q}I$ but $I^3 = \mathfrak{q}I^2$.*

Proof. We have $J = Q + (x_ix_m \mid 1 \leq i \leq m-1) + (v)$ (cf. Proof of Lemma (4.1)). Assume that $J^2 = QJ$. Then $v^2 \in QJ$ whence

$$V^2 \in (A_i \mid 1 \leq i \leq d)\cdot [(A_i \mid 1 \leq i \leq d) + (X_iX_m \mid 1 \leq i \leq m-1) + (V)] + \mathfrak{a}. \tag{4.8}$$

We now substitute $X_i = 0$ and $A_j = 0$ for all $2 \le i \le m$ and $2 \le j \le d$. Then by (4.8) we get

$$V^2 \in (A_1^2, A_1V, X_1^2, X_1V, V^2 - A_1X_1)$$

in the polynomial ring $k[X_1, V, A_1]$, which is impossible. Hence $v^2 \notin QJ$ so that we have $J^2 \neq QJ$. We get $J^3 = QJ^2$, because $J^2 = QJ + (v^2)$ and $v^3 = 0$. $\square$

Therefore, for given integers $1 \le d < m$, there exists a Buchsbaum local ring A with $\dim A = d$, $\operatorname{depth} A = d - 1$, and $\mathrm{e}(A) = 2m$, such that A contains a parameter ideal Q which is a minimal reduction of $\mathfrak{m}$ and $I^2 \neq QI$ but $I^3 = QI^2$, where $I = Q : \mathfrak{m}$. Thus the equality $I^2 = QI$ fails in general to hold, even though A is a Buchsbaum local ring with sufficiently large depth and multiplicity.

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